MATH 3060 Assignment 5 solution

Chan Ki Fung

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1. We will define F by (For $x_n \in E$)

$$F(\lim x_n) = \lim f(x_n)$$

We need to check

(a) $\lim f(x_n)$ exists if $\lim x_n$ exists.
(b) If $\lim x_n = \lim x'_n$, then $\lim f(x_n) = \lim f(x'_n)$.
(c) If $x = \lim x_n \in E$, then $\lim f(x_n) = f(x)$.
(d) F is uniformly continuous.

To show (a), we need to check $f(x_n)$ is Cauchy. In fact, let $\epsilon > 0$, we can find $\delta > 0$ such that for any $x, x' \in E$,

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon.$$

So it suffices to show that we can find an integer N such that for $n, m > N$

$$d_X(x_n, x_m) < \delta,$$

and this is true because (x_n) is Cauchy.
For (b), note that $d_X(x_n, x'_n) \to 0$, so

$$d_Y(\lim f(x_n), \lim f(x'_n)) = \lim d_Y(f(x_n), f(x'_n)) = 0$$

because f is uniformly continuous.
(c) follows from the definition of f being continuous.
(d) Let ϵ, we can find $\delta > 0$ such that for $x, x' \in E$

$$d_X(x, x') < 3\delta \implies d_Y(f(x), f(x')) < \epsilon.$$

Now let $x, x' \in \overline{E}$, and choose $\{x_n\}, \{x'_n\} \subset E$ with $x = \lim x_n, y = \lim x'_n$. Then for n sufficiently large, we have

$$d_X(x_n, x) < \delta, d_X(x'_n, x') < \delta.$$

We must have $d_X(x_n, x'_n) < 3\delta$, and hence

$$d_Y(F(x), F(x')) = d_Y(\lim f(x_n), \lim f(x'_n)) = \lim d_Y(f(x_n), f(x'_n)) < \epsilon.$$

2. We write $x - 3x\sin x + x^4 = \Phi = \mathrm{Id}(x) + \Psi(x)$, where $\Psi(x) = -3x\sin x + x^4$. For $|x|, |x'| < r < 1$, we have, by mean value theorem,

$$|\Psi(x) - \Psi(x')| = |(-3\sin\xi - 3\xi\cos\xi + 4\xi^3)||x - x'| \leq 10r|x - x'|.$$

We will conclude by perturbation of identity that we can choose r so that $0.001 \in \mathrm{Im}(\Phi)$. To do this, we need $10r < 1$, and $(1 - 10r)r > 0.001$. This can be done, for example taking $r = 0.09$.

3. We write $\Phi = \mathrm{Id} + \Psi$, where $\Psi(x, y) = (y^4, -x^2)$. For $p = (x, y), p' = (x', y') \in B_0(r)$, we have

$$\begin{aligned} ||\Psi(p) - \Psi(p')||^2 &= (y^4 - y'^4)^2 + (x^2 - x'^2)^2 \\ &\leq (4\xi_y^3)^2(y - y')^2 + (2\xi_x)^2(x - x')^2 \\ &\leq 16r^6||p - p'||^2 + 4r^4|p - p'|^2 \end{aligned}$$

If we assume $r < 1$, then we have

$$||\Psi(p) - \Psi(p')|| \leq \sqrt{20r^2}||p - p'||^2 \leq 5r||p - p'||.$$

We then apply the Perturbation of identity, so we need to choose r so that $5r < 1$ and $(1 - 5r)r > 0.01$. We can take, for example $r = 0.1$.

4. Note that $(I - A)^t = I - A^t$, so it suffices to show $I - A^t$ is invertible. The idea is to show that A^t is a contraction. If this is true, then $I - A^t$ must be invertible. This is because if $x \neq 0$ and $(I - A^t)x = 0$, then $|x| > |A^t x| = |-x| = |x|$, which is a contradiction.
However, A^t may not be a contraction for the standard metric, but it is a contraction for d_{sup} and d_1. We will do the case for d_1.
Take $\gamma = \max_i \sum_j |a_{ij}| < 1$. We need to show that $|A^t x|_1 \leq \gamma |x|_1$. But

$$\begin{aligned} |A^t x|_1 &= \sum_j \sum_i |a_{ij} x_i| \\ &\leq \sum_i \left(|x_i| \sum_j |a_{ij}| \right) \\ &\leq \gamma \sum_i |x_i| \\ &= \gamma |x|_1 \end{aligned}$$